

A TIGHTER ERDŐS-PÓSA FUNCTION FOR LONG CYCLES

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ABSTRACT. We prove that there exists a bivariate function f with $f(k, \ell) = O(\ell \cdot k \log k)$ such that for every naturals k and ℓ , every graph G has at least k vertex-disjoint cycles of length at least ℓ or a set of at most $f(k, \ell)$ vertices that meets all cycles of length at least ℓ . This improves a result by Birmelé, Bondy and Reed (Combinatorica, 2007), who proved the same result with $f(k, \ell) = \Theta(\ell \cdot k^2)$.

1. INTRODUCTION

A collection of graphs \mathcal{H} is said to have the *Erdős-Pósa* property if there exists a function $f : \mathbb{N} \rightarrow \mathbb{R}_+$ such that for every natural k and every graph G at least one of the following two assertions holds:

- G contains a collection of k vertex-disjoint subgraphs G_1, \dots, G_k , each isomorphic to a graph in \mathcal{H} ;
- G contains a set X of $f(k)$ vertices such that no subgraph of $G - X$ is isomorphic to a graph in \mathcal{H} .

A collection G_1, \dots, G_k as above is called a *packing* and a set X as above is called a *transversal*. These definitions are motivated by a celebrated result of Erdős and Pósa [3]. Denoting by C_t the cycle of length t , they proved that $\mathcal{H} = \{C_t \mid t \geq 3\}$ has the Erdős-Pósa property. They obtain a function f in $\Theta(k \log k)$ and prove that this function f is best possible, up to a constant.

Our main result is as follows.

Theorem 1.1. *There exists a function $f : \mathbb{N}^2 \rightarrow \mathbb{R}_+$ with $f(k, \ell) = O(\ell \cdot k \log k)$ such that for every $k, \ell \in \mathbb{N}$ and every graph G , at least one of the two following assertions holds:*

- G contains k vertex-disjoint cycles of length at least ℓ ;
- G contains a set of $f(k, \ell)$ vertices meeting all the cycles of length at least ℓ .

This result implies in particular that the collection $\mathcal{H} := \{C_t \mid t \geq \ell\}$ has the Erdős-Pósa property for each fixed natural ℓ . Birmelé, Bondy and Reed [1] proved Theorem 1.1 with $f(k, \ell) = \Theta(\ell \cdot k^2)$ and left as an open problem to find the correct order of magnitude of f . Theorem 1.1 essentially settles this problem. Our function f is best possible up to a constant for each fixed ℓ . Moreover, it is also best possible up to a constant for each fixed k . However, we do not know whether it is best possible up to a constant when both k and ℓ vary.

Before giving the outline of this paper, we mention a few relevant references concerning the case where \mathcal{H} consists of all the graphs containing a fixed graph H as a

minor. Robertson and Seymour have shown that \mathcal{H} has the Erdős-Pósa property if and only if H is planar [5]. They left wide open the problem of determining the order of magnitude of the best possible function f , for each fixed planar graph H . Our main result answers this problem when H is a cycle. A recent paper of Fiorini, Joret and Wood [4] answers the problem when H is a forest. In this case, it turns out that f can be taken to be linear in k .

The outline of the rest of the paper is as follows. We begin with some preliminaries in Section 2. The proof of Theorem 1.1 is given in Section 3.

2. PRELIMINARIES

Before proving our main result, we state a few lemmas that are used in the proof. For $k \in \mathbb{N}$, let

$$s_k := \begin{cases} 4k \log k + 4k \log \log k + 16k & \text{if } k \geq 2 \\ 1 & \text{if } k \leq 1. \end{cases}$$

Notice that $s_k = \Theta(k \log k)$.

Lemma 2.1 (Erdős and Pósa [3], Diestel [2]). *For every $k \in \mathbb{N}$, every cubic multigraph H with at least s_k vertices contains k vertex-disjoint cycles.*

Let $\ell \in \mathbb{N}$ be fixed. Below, we call a cycle *long* if its length is at least ℓ , and *short* otherwise. Our proof relies on the following lemma (see below). Birmelé, Bondy and Reed conjecture that the lemma still holds when $2\ell + 3$ is replaced by ℓ , which would be tight.

Lemma 2.2 (Birmelé, Bondy and Reed [1]). *If a graph G does not contain two vertex-disjoint long cycles, then it contains a set of at most $2\ell + 3$ vertices that meets all the long cycles.*

Compared to the two previous lemmas, our next lemma is rather obvious. We nevertheless include a proof for completeness.

Lemma 2.3. *Let z and z' be two distinct vertices of G . Let C_z and $C_{z'}$ denote two cycles of G of length at least 2ℓ containing z and z' , respectively. If C_z and $C_{z'}$ are not disjoint, then $C_z \cup C_{z'}$ contains a z - z' path of length at least ℓ .*

Proof. Follow C_z in any direction from z until the first vertex of $C_{z'}$ is reached, say t . One of the two t - z' paths in $C_{z'}$ has length at least ℓ . Thus the desired z - z' path exists. \square

3. THE PROOF

Proof of Theorem 1.1. We prove the theorem with $f(k) := (2\ell + 4)(k - 1) + (3\ell/2 + 1)s_k$, by induction on k . For $k \leq 1$, the theorem obviously holds. From now on, we assume $k \geq 2$. Because $f(k - 1) + 2\ell \leq f(k)$, we may assume without loss of generality that G does not contain any cycle of length between ℓ and 2ℓ .

Let H denote a subgraph of G with the following properties:

- (i) all vertices v of H have degree 2 or 3 in H ;
- (ii) H contains no short cycle;
- (iii) the size of $U := \{v \in V(H) \mid \deg_H(v) = 3\}$ is maximum.

Notice that all components of $H - U$ are either cycles or paths. Among all subgraphs H satisfying (i)–(iii), we choose one such that the number of cycles in $H - U$ is maximum.

Let U' denote the set of vertices of H whose distance in H to U is at most $\ell/2$. In a formula,

$$U' := \{v \in V(H) \mid d_H(U, v) \leq \ell/2\}.$$

Consider a H -path P that avoids U' . We claim that the endpoints of P are in the same component of $H - U$. Suppose that P has its endpoints in two different components of $H - U$. If one of these components is a cycle, then $H + P$ satisfies (i)–(ii) but has two more degree-3 vertices, a contradiction. If the two components are paths, then $H + P$ always satisfies (i), and also satisfies (ii) unless a short cycle appears when P is added to H . This short cycle intersects U . In particular, one of the two endpoints of P is at distance at most $\ell/2$ from U . Hence P is not disjoint from U' , in contradiction with our hypotheses. Therefore, our claim holds.

Now consider a long cycle C that avoids U' .

By choice of H , this cycle C has some vertex in H , because otherwise we could add C to H and increase the number of cycles in $H - U$ without changing the size of U , contradicting our choice of H .

It could well be that C meets H in exactly one vertex. For now, we assume that C contains at least two vertices of H . By the above claim, C meets exactly one component of $H - U$, say K . Let K' denote the subgraph of G obtained by adding to K all the H -paths P with both endpoints in K . Then C is contained in K' .

Case 1. K is a cycle. We claim that K' does not contain two vertex-disjoint long cycles. Otherwise, we could redefine H by replacing K by these two cycles and contradict the choice of H . By Lemma 2.2, K' contains a set of at most $2\ell + 3$ vertices that meets all the long cycles in K' .

Case 2. K is a path. We claim that this case cannot occur. Indeed, let u and v denote the two endpoints of K . We can redefine H by replacing K by the subgraph of K' obtained from the long cycle C by connecting u and v to this cycle through K . Because two of the vertices of K are in C , and C avoids U' , this operation preserves properties (i) and (ii). But this operation increases the number of vertices in U , and thus contradicts our choice of H .

Now to the conclusion. Let \mathcal{C} denote the set of long cycles that avoid U and meet H in exactly one vertex. Let Z denote the set of vertices of H that are in some cycle of \mathcal{C} . For each vertex $z \in Z$, pick a *witness* cycle C_z in \mathcal{C} . Any two distinct witness cycles C_z and $C_{z'}$ are disjoint because otherwise, by Lemma 2.3, we could find in their union a H -path with endpoints z and z' of length at least ℓ . Adding such an

H -path to H does not create a short cycle and increases the number of vertices in U , a contradiction. Therefore,

$$\mathcal{C}' := \{C_z \mid z \in Z\}$$

is a collection of vertex-disjoint long cycles.

Let \mathcal{D} denote the set of cycles in $H - U$ that are disjoint from Z . Thus, $\mathcal{C}' \cup \mathcal{D}$ is also a collection of vertex-disjoint long cycles.

If $\mathcal{C}' \cup \mathcal{D}$ contains at least k cycles, then the theorem holds. Assume now that $|\mathcal{C}' \cup \mathcal{D}| \leq k - 1$.

Now that we have failed to produce a large packing of long cycles, we try for a small transversal of the long cycles. The transversal is obtained as follows:

- in each component K of $H - U$ that is a cycle, pick a set of at most $2\ell + 3$ vertices that meet all the long cycles in K' (by Lemma 2.2, such a set exists);
- add all the vertices of Z ;
- add all the vertices of U' .

The total number of vertices in the transversal is bounded by

$$\begin{aligned} (2\ell + 3)(|\mathcal{C}' \cup \mathcal{D}|) + |Z| + |U'| &\leq (2\ell + 4)(k - 1) + |U'| \\ &\leq (2\ell + 4)(k - 1) + (3\ell/2 + 1)|U|. \end{aligned}$$

If $|U| \leq s_k$ then the theorem holds. Otherwise, we have $|U| \geq s_k$ and by Lemma 2.1 applied to the multigraph obtained from H by suppressing all degree-2 vertices, H contains k vertex-disjoint cycles. By choice of H , these cycles are all long. Thus the theorem holds in this case also. \square

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